Models of Set Theory II - Winter 2013

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht Problem sheet 9

Problem 32 (4 Points)**.** Suppose that $\mathcal{I}$ is an ideal on the set $Borel(\mathbb{R})$ of Borel subsets of $\mathbb{R}$.

(a) Check that the inclusion on $Borel(\mathbb{R})$ induces a partial order on $Borel(\mathbb{R})/\mathcal{I}$.
(b) Show that $Borel(\mathbb{R})/\mathcal{M}$ and $Borel(\mathbb{R})/\mathcal{N}$ are c.c.c.

Problem 33 (4 Points)**.** Suppose that B is a σ-complete Boolean algebra, i.e. B is a Boolean algebra such that infima and suprema of countable sets exist. Suppose that $\mathcal{I}$ is a σ-ideal on B, i.e. $\mathcal{I}$ is downwards closed, $0 \in \mathcal{I}$, $1 \notin I$, and $\mathcal{I}$ is closed under suprema of countable sets.

(a) Check that $B/\mathcal{I}$ is a Boolean algebra.
(b) If $B/\mathcal{I}$ is c.c.c., show that $B/\mathcal{I}$ is a complete Boolean algebra.

Problem 34 (2 Points)**.** (a) Show that $Borel(\mathbb{R})/\mathcal{M}$ has a countable dense subset and hence there is a dense embedding $f \colon \mathbb{P} \to Borel(\mathbb{R})/\mathcal{M}$, where $\mathbb{P}$ denotes Cohen forcing.
(b) Check that there is a dense embedding $g \colon \mathbb{Q} \to Borel(\mathbb{R})/\mathcal{N}$, where $\mathbb{Q}$ denotes random forcing.

Problem 35 (8 Points)**.** Suppose that M is a transitive model of ZFC. A set $S \subseteq \mathbb{R}$ is *Solovay over M* if there is a formula $\varphi(x, \vec{y})$ and $\vec{a} \in M$ such that for all $x \in \mathbb{R}$ such that x codes an M-generic filter for some forcing $P \in M$:

$$x \in S \iff M[x] \vDash \varphi(x, \vec{a}).$$

Now suppose that S is Solovay over M.

Let P denote $Borel(\mathbb{R})$ with $A \leq_P B :\iff A \setminus B \in \mathcal{M}$. Let $C(M)$ denote the set of $(P^*)^M$-generic reals (i.e. Cohen reals) x over M, i.e. such that

$$\{x\} = \bigcap \{[a,b]^{M[G]} \mid a,b \in \mathbb{Q},\ a < b,\ [a,b]^M \in G\}$$

for some M-generic filter G for P^M.

Let Q denote $Borel(\mathbb{R})$ with $A \leq_{\mathbb{Q}} B :\iff A \setminus B \in \mathcal{N}$. Let $R(M)$ denote the set of $(Q^*)^M$-generic reals x (i.e. random reals) over M, i.e. such that

$$\{x\} = \bigcap \{d^{M[G]} \mid d \text{ is an } F\text{−code in } M \text{ and } d^M \in G\}$$

for some M-generic filter G for Q^M.

(a) Show that there is a Borel set $A \subseteq \mathbb{R}$ with $S \cap C(M) = A \cap C(M)$.
(*Hint: Find an F_σ set A with $[A]_{\mathcal{M}} = [\![\varphi(\dot{x})]\!]_P$, where $\dot{x}$ is a $(P^*)^M$-name in M for the $(P^*)^M$-generic real, and apply the forcing theorem over M. An F_σ set A is of the form $A = \bigcup_{n\in\omega} A_n$, where each A_n is closed)*

(b) Conclude that S has the property of Baire if $\mathbb{R} \setminus C(M) \in \mathcal{M}$.

(c) Show that there is a Borel set $A \subseteq \mathbb{R}$ with $S \cap R(M) = A \cap R(M)$ (*Hint: Find an* F_σ *set* A *with* $[A]_{\mathcal{N}} = [\![\varphi(\dot{x})]\!]_Q$, *where* $\dot{x}$ *is a* $(Q^*)^M$*-name in* M *for the* $(Q^*)^M$*-generic real.)*

(d) Conclude that S is Lebesgue measurable if $\mathbb{R} \setminus R(M) \in \mathcal{N}$.

Please hand in your solutions on Monday, January 06 before the lecture.